Asymptotically isochronous systems

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Abstract

Mechanisms are elucidated underlying the existence of dynamical systems whose *generic* solutions approach *asymptotically* (at large time) *isochronous* evolutions: *all* their dependent variables tend *asymptotically* to functions *periodic* with the *same* fixed period. We focus on two such mechanisms, emphasizing their generality and illustrating each of them via a representative example. The first example belongs to a recently discovered class of *integrable* indeed *solvable* many-body problems. The second example consists of a broad class of (generally *nonintegrable*) models obtained by deforming appropriately the well-known (*integrable* and *isochronous*) many-body problem with inverse-cube two-body forces and a one-body linear ("harmonic oscillator") force.

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1 Introduction

Over the last three-four decades major progress occurred in the discovery and understanding of *integrable* dynamical systems with a finite or infinite number of degrees of freedom, and over the last decade the possibility was noticed and exploited to identify and investigate many *isochronous* dynamical systems

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characterized by a time evolution *completely periodic* (i. e., periodic in *all* degrees of freedom) with the *same* period. This *isochronous* evolution might prevail in the *entire* (natural) phase space of the model under consideration (one talks then of an *entirely isochronous* system), implying of course that such a model is certainly *integrable*; or it might only prevail in an *open* (hence fully dimensional) region of its (natural) phase space, a phenomenology now known to characterize large families of *nonintegrable* dynamical systems possibly featuring quite complicated ("chaotic") behaviors outside the *isochronous* phase space region (for a review of these developments, see $[1, 2]$). In the present paper we discuss another, perhaps more interesting, phenomenology, namely dynamical systems whose *generic* solutions approach *asymptotically* (at large time) *isochronous* evolutions: *all* their dependent variables tend *asymptotically* to functions *periodic* with the *same* fixed period. The *definition* of such dynamical systems is provided by the simultaneous validity of the two formulas

$$\lim_{t\rightarrow+\infty}\left[z_n\left(t\right)-\tilde{z}_n\left(t\right)\right]=0\ ,\quad n=1,...,N\ , \tag{1a}$$

$$\tilde{z}_n\left(t+\tilde{T}\right)=\tilde{z}_n\left(t\right)\ ,\quad n=1,...,N\ . \tag{1b}$$

Notation: the N (generally *complex*; but see below) numbers $z_n\left(t\right)$ denote the N dependent variables of the dynamical system under consideration; we restrict consideration to the case when N is a *finite* positive integer; the *real* variable t denotes the time; the N functions $\tilde{z}_n\left(t\right)$ characterize the asymptotic behavior of the dynamical system via (1a) and the periodicity requirement (1b) they satisfy characterizes the property of *asymptotic isochronicity.* This property is supposed to hold in an *open* (hence fully dimensional) region of the phase space of the dynamical system under consideration (possibly coinciding with its entire natural phase space): hence the dependent variables $z_n\left(t\right)$ denote here (the N components of) a *generic* solution of the dynamical system evolving (at least for sufficiently large time) within that region, while the functions $\tilde{z}_n\left(t\right)$, which shall generally be different for different solutions $z_n\left(t\right)$, are required to satisfy the periodicity property (1b) with the *fixed* period $\tilde{T}$ (the *same* for all the solutions in the phase space region under consideration). Of course the formula (1a) does not define uniquely – for a given N-vector $\underline{z}\left(t\right)$ – a corresponding N-vector $\underline{\tilde{z}}\left(t\right)$: the time-dependent N-vector $\underline{\tilde{z}}\left(t\right)$ is only identified by (1a) up to arbitrary corrections whose effects disappear in the asymptotic limit $t\rightarrow\infty$. The property of *asymptotic isochronicity* is guaranteed provided there exist just one N-vector $\underline{\tilde{z}}\left(t\right)$ satisfying *both* relations (1), for every *generic* solution $\underline{z}\left(t\right)$ in an *open*, fully dimensional, region of phase space – namely for every solution $\underline{z}\left(t\right)$ in that region of phase space, except possibly for some *exceptional*, generally *singular*, solutions belonging to a *lower dimensional* sector of that phase space region.

The elementary idea underlying the identification of large classes of such *asymptotically isochronous* dynamical systems is to start from *isochronous* systems and then modify them by introducing a deformation whose effects are significant through the time evolution yet disappear at large time: so that the modified systems loose their *isochronous* character (at finite times) but in some sense retain it (at large times) as the dominant feature characterizing their *asymptotic* behavior.

There are several possible ways to implement this strategy in order to manufacture *asymptotically isochronous* systems: some are rather trivial, some less so. This kind of judgement is of course subjective: for instance we tend to think that an important requirement for such systems to be deemed "interesting" is that they be *autonomous* – because the interest of dynamical systems is also related to their potential usefulness in order to model natural phenomena, which are generally described by *autonomous* evolution equations – and moreover because the freedom to introduce instead an *explicit* time dependence in the equations of motion of a dynamical system would provide too easy a way to influence more or less at will the asymptotic behavior of such a system. But of course the difference between *autonomous* and *nonautonomous* systems is unessential, since any *nonautonomous* system can be made *autonomous* by treating time itself as an additional dependent variable.

In this paper we focus on two mechanisms yielding (autonomous) *asymptotically isochronous* systems, and illustrate each of them via a representative example. The first example (see Section 2) belongs to a recently discovered class of *integrable* indeed *solvable* many-body problems [3]; in this case we eventually focus on as simple and specific an example as possible, which is also suitable to exhibit some numerical results – but we trust our presentation is adequate to illustrate the generality of the approach. In this case the periodic behavior prevailing asymptotically corresponds to a special solution of the dynamical system under consideration belonging to a region of phase space with *positive* codimension – albeit *not* an *isolated* solution of this system, so not quite identifiable as a *limit cycle*. Hence this model might be considered a representative example of a phenomenology characterized by the presence of some kind of friction. The second example (see Section 3) consists of a broad class of models obtained by deforming appropriately the well-known (see for instance [4]) *integrable* and *isochronous* one-dimensional many-body problem with inverse-cube two-body forces and a one-body linear ("harmonic oscillator") force; the alert reader will again appreciate the generality of the approach, even though we illustrate it by focusing on a specific model (also restricting consideration to *real* dependent variables). In this second case the time-dependent N-vector to which the solutions of the model tend *asymptotically* is *not* restricted to be in a sector of phase space with *positive* codimension and is generally *not* itself a solution of the *asymptotically isochronous* N-body model, so this phenomenology does not correspond to what is generally referred to as a *limit cycle* behavior. In each of these two cases we back the qualitative understanding of the origin of the relevant phenomenology with a *proof* of its actual emergence, see (1). A section entitled "Outlook" in which we elaborate tersely on the generality of this phenomenology concludes the paper.

2 An asymptotically isochronous class of solvable many-body problems

A particular mechanism to manufacture *integrable*, indeed *solvable*, dynamical systems interpretable as many-body problems inasmuch as they are characterized by Newtonian equations of motion ("acceleration equal force") was introduced about three decades ago [5] and has been subsequently exploited to identify and investigate several such systems (for reviews of these developments see for instance [2, 4]). The idea is to exploit the *nonlinear* relation among the N coefficients $c_m(t)$ of a (for definiteness, monic) time-dependent polynomial of degree N and its N zeros $z_n(t)$:

$$\psi(z,t) = z^N + \sum_{m=1}^{N} c_m(t)\, z^{N-m} = \prod_{n=1}^{N} \left[z - z_n(t) \right] \ . \tag{2}$$

A class of such systems is characterized by the fact that the N coefficients $c_m(t)$ evolve in time according to a system of *linear* second-order *constant-coefficient* ODEs, the solution of which is a purely algebraic task (requiring essentially the diagonalization of an explicitly known matrix of order N). The determination of the corresponding time evolution of the N zeros $z_n(t)$ is therefore as well a purely algebraic task: computing the N zeros of a known polynomial. And it so happens that in many cases [2, 4, 5] this time evolution is indeed interpretable as that characterizing a Newtonian N-body problem – hence a *solvable* N-body problem, since its solution can be achieved by purely algebraic means.

Indeed the solution $z_n(t)$ of such a model is reduced to finding the N zeros of a polynomial of degree N in the (*complex*) variable z, see (2), whose coefficients $c_m(t)$ generally evolve exponentially in time, typically

$$c_m(t) = \sum_{\ell=1}^{N} \left\{ \gamma^{(\ell,+)} u_m^{(\ell,+)} \exp\left[\lambda^{(\ell,+)} t\right] + \gamma^{(\ell,-)} u_m^{(\ell,-)} \exp\left[\lambda^{(\ell,-)} t\right] \right\} , \tag{3}$$

where the $2N$ constants $\gamma^{(\ell,\pm)}$ are arbitrary (to be determined by the initial data $z_n(0), \dot{z}_n(0)$ in the context of the initial-value problem for the N-body system) and the $2N$ numbers $\lambda^{(\ell,\pm)}$ respectively the quantities $u_m^{(\ell,\pm)}$ are the eigenvalues respectively the (components of the) eigenvectors of the matrix eigenvalue problem characterizing, as explained above, the dynamics of this system. Note that these eigenvalues and eigenvectors are associated to the dynamical problem under consideration: they do *not* depend on the initial data identifying a particular solution, namely they are the *same* for all the solutions of the system.

It is now clear (and indeed well known [2, 4, 5]) that if the $2N$ eigenvalues $\lambda^{(\ell,\pm)}$ are all *integer* multiples of a single *imaginary* number $i\omega$ (with $\omega > 0$), $\lambda^{(\ell,\pm)} = ik^{(\ell,\pm)}\omega$ with the $2N$ numbers $k_\ell^{(\pm)}$ arbitrary *integers* (positive or negative, but not vanishing), then the polynomial $\psi(z,t)$ is clearly *periodic* with the (possibly nonprimitive) period

$$T = \frac{2\pi}{\omega} \ , \tag{4a}$$

$$\psi(z, t+T) = \psi(z,t) \ , \tag{4b}$$

hence all its zeros $z_n(t)$ are as well *periodic* with this same period or possibly with a (generally small [6]) *integer* multiple p of this period, $\tilde{T} = pT$, due to the possibility that they exchange their role through the time evolution. Hence the corresponding N-body problem is *isochronous.*

And it is as well plain that if, out of the $2N$ eigenvalues $\lambda^{(\ell,\pm)}$, only a (nonempty) subset have the property indicated above while *all* the others feature a *negative* real part, then the many-body problem in question is *asymptotically isochronous.* This observation is not new, see for instance Section 4.2.3 of Ref. [4] (entitled "Some special cases: models with a limit cycle, models with confined and periodic motions, Hamiltonian models, translation-invariant models, models featuring equilibrium and spiraling configurations, models featuring only completely periodic motions"); but, to the best of our knowledge, this mechanism yielding *asymptotically isochronous* many-body problems was never analyzed in explicit detail (including the display of numerical results). This is what we do in this section, by focusing on a specific model whose *integrable*, indeed *solvable*, character has been ascertained only quite recently [3].

2.1 A specific example

This N-body problem (with $N \geq 3$) is characterized by the Newtonian equations of motion

$$\ddot{z}_n = -a_1\dot{z}_n + a_2 z_n \frac{z_n^2 - 5}{z_n^2 - 1} - 2a_3 \frac{z_n^2 + 1}{z_n^2 - 1} - 2a_4 z_n$$
$$+2 \sum_{m=1, m\neq n}^{N} \frac{\dot{z}_n \dot{z}_m + a_2 + a_3 z_n + a_4\left(z_n^2 - 1\right)}{z_n - z_m} \ , \quad n = 1, ..., N \ , \tag{5a}$$

where the 4 "coupling constants" a_j are *a priori arbitrary* complex numbers, superimposed dots denote time-differentiations and the rest of the notation is self-evident. The *solvable* character of this N-body problem hinges [3] upon the following 4 restrictions on its *initial* data:

$$\sum_{n=1}^{N} \frac{1}{z_n(0) \pm 1} = 0 \ , \quad \sum_{n=1}^{N} \frac{\dot{z}_n(0)}{\left[z_n(0) \pm 1\right]^2} = 0 \ , \tag{5b}$$

which are then sufficient [3] to guarantee that, throughout the time evolution,

$$\sum_{n=1}^{N} \frac{1}{z_n(t) \pm 1} = 0 \ , \tag{5c}$$

implying that for this model it is justified to assume that only the evolution of $N-2$ particles is determined by the Newtonian equations of motion (5a), while the evolution of the remaining two is determined by these conditions, see (5c).

Then the evolution of the N "particle coordinates" $z_n(t)$ – taking generally place in the *complex* z-plane – coincides with the evolution of the N zeros of a monic polynomial of degree N in the variable z analogous to $\psi(z,t)$, see (2), but more specifically reading as follows [3]:

$$\psi(z,t) = \pi_N(z) + \sum_{m=1}^{N-3} \left[c_m(t)\, \pi_{N-m}(z)\right] + c_N(t) \ , \tag{6a}$$

$$\pi_m(z) = z^m - \varepsilon_m \frac{m}{2} z^2 - \varepsilon_{m+1} m z, \quad m = 0, 1, ..., N \ , \tag{6b}$$

$$\varepsilon_m = 1 \text{ if } m \text{ is } \textit{even} \ , \quad \varepsilon_m = 0 \text{ if } m \text{ is } \textit{odd} \ . \tag{6c}$$

And the coefficients $c_m(t)$ evolve indeed according to formulas analogous to (3), but more specifically reading as follows [3]:

$$c_m(t) = \sum_{\ell=1, \ell \neq N-1, N-2}^{N} \left\{ \gamma^{(\ell,+)} u_m^{(\ell,+)} \exp\left[\lambda^{(\ell,+)} t\right] + \gamma^{(\ell,-)} u_m^{(\ell,-)} \exp\left[\lambda^{(\ell,-)} t\right] \right\} \ ,$$
$$m = 1, ..., N-3 \ \text{ and } \ m = N \ , \tag{7a}$$

$$\lambda^{(\ell,\pm)} = \frac{-a_1 \pm \Delta_\ell}{2} \ , \quad \Delta_\ell^2 = a_1^2 + 4\ell \left[a_2 + (2N - \ell - 3)\, a_4\right] \ ,$$
$$\ell = 1, ..., N-3 \ , N \ . \tag{7b}$$

Note that the coupling constant a_3 does not appear explicitly in these formulas, but of course all 4 coupling constants a_j do play a role in determining the quantities $u_m^{(\ell,\pm)}$ appearing in the right-hand side of (7a).

We now restrict attention to the $N = 3$ case, since this is sufficient, indeed convenient, for exhibiting quite explicitly an *asymptotically isochronous* model. Then the only relevant coefficient (see (7a)) is

$$c_3\left(t\right) = \gamma_+ \exp\left(\lambda_+ t\right) + \gamma_- \exp\left(\lambda_- t\right) \ , \tag{8a}$$

$$\lambda_\pm = \frac{-a_1 \pm \Delta}{2} \ , \quad \Delta^2 = a_1^2 + 12 a_2 \ , \tag{8b}$$

where the somewhat simplified notation we are now using is we trust self-explanatory (and note that in this case with $N = 3$ the eigenvalues $\lambda_\pm$ only depend on the two coupling constants a_1 and a_2). Correspondingly, the positions of the 3 moving particles are the 3 zeros $z_n\left(t\right)$ of the third-degree polynomial

$$\psi\left(z,t\right) = \pi_3\left(z\right) + c_3\left(t\right) = z^3 - 3z + c_3\left(t\right) = \prod_{n=1}^{3}\left[z - z_n\left(t\right)\right] \ . \tag{8c}$$

Note that these 3 zeros automatically satisfy the requirements (5c), which corresponds [3] to the condition that the partial derivative of $\psi\left(z,t\right)$ with respect to z vanish at $z = \pm 1$, $\psi_z\left(\pm 1, t\right) = 0$.

Assume now that the two coupling constants a_1 and a_2 entail, via (8b),

$$\lambda_+ = i\omega \ , \quad \lambda_- = -\alpha + i\beta \ , \tag{9a}$$

with α *positive*, $\alpha > 0$, ω also *positive*, $\omega > 0$ (for definiteness), and β *real* but otherwise *arbitrary*. This indeed happens provided

$$a_1 = \alpha - i\left(\beta + \omega\right) \ , \quad a_2 = \frac{\omega\left(\beta + i\alpha\right)}{3} \ . \tag{10}$$

It is now plain that the asymptotic condition (1a) holds now with $\tilde{z}_n\left(t\right)$ being the three roots of the polynomial $z^3 - 3z + \gamma_+ \exp\left(i\omega t\right)$,

$$z^3 - 3z + \gamma_+ \exp\left(i\omega t\right) = \prod_{n=1}^{3}\left[z - \tilde{z}_n\left(t\right)\right] \ , \tag{11}$$

which provide of course also the special solution of the model (5) (with $N = 3$) corresponding to initial data such that γ_- vanishes (see (8a)). And it is as well plain that the time evolution of this polynomial is periodic with period T, see (4a), hence the corresponding evolution of each of its 3 zeros is clearly periodic with periods T, $2T$ or $3T$, depending whether that zero does not "exchange its role" through the motion with another zero or does so with one or with both the other two zeros.

We complete this section by displaying one specific example, namely the solution of the system of ODEs (5) with $N = 3$, $\omega = 2\pi$ implying $T = 1$ (see

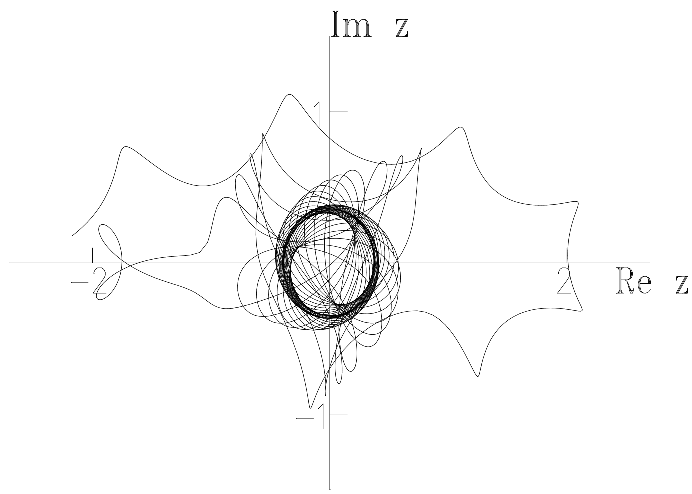

Fig. 1. Trajectory of $z_1(t)$ in the complex z-plane from $t = 0$ to $t = 50$ (see text)

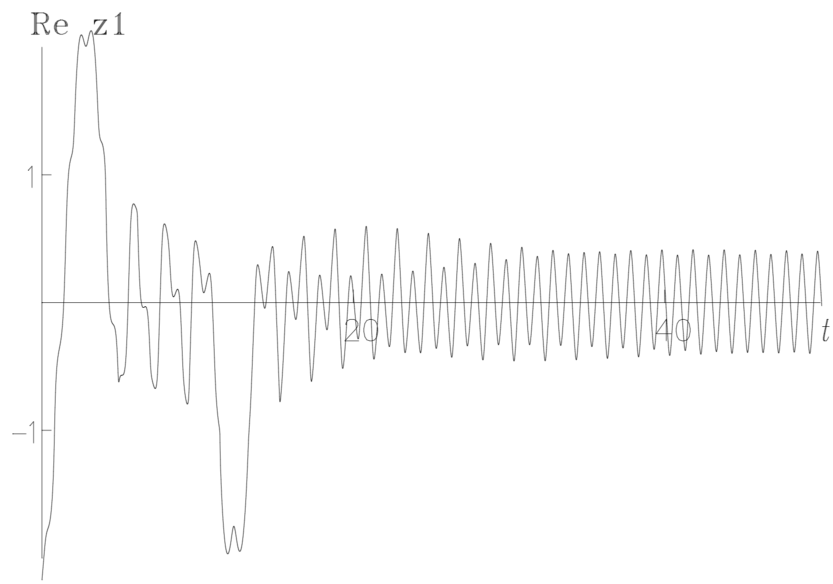

Fig. 2. Plot of $\mathrm{Re}\, z_1$ as a function of t (see text)

(4a)), $a_3 = a_4 = 0$, a_1 and a_2 given by (10) with $\alpha = 0.1$ and $\beta = -3$, and with initial data

$$
\begin{aligned}
z_1(0) &= -2.1702823 + 0.18021431\mathrm{i}, & \dot{z}_1(0) &= 1.2487698 + 0.76941297\mathrm{i}, \\
z_2(0) &= 0.71910399 - 0.89149288\mathrm{i}, & \dot{z}_2(0) &= -2.7507203 + 1.3102500\mathrm{i}, \\
z_3(0) &= 1.4511783 + 0.71127857\mathrm{i}, & \dot{z}_3(0) &= 1.5019505 - 2.0796630\mathrm{i},
\end{aligned}
$$

satisfying the conditions (5b) and entailing $\gamma_+ = 0.5 + i$, $\gamma_- = 3 - 3i$ (see (8a)). The results displayed are, from $t = 0$ to $t = 50$, the trajectory of $z_1(t)$ in the complex z-plane (Fig. 1), the real part of $z_1(t)$ as a function of t (Fig. 2) (the behavior of the imaginary part is qualitatively analogous) and the evolution

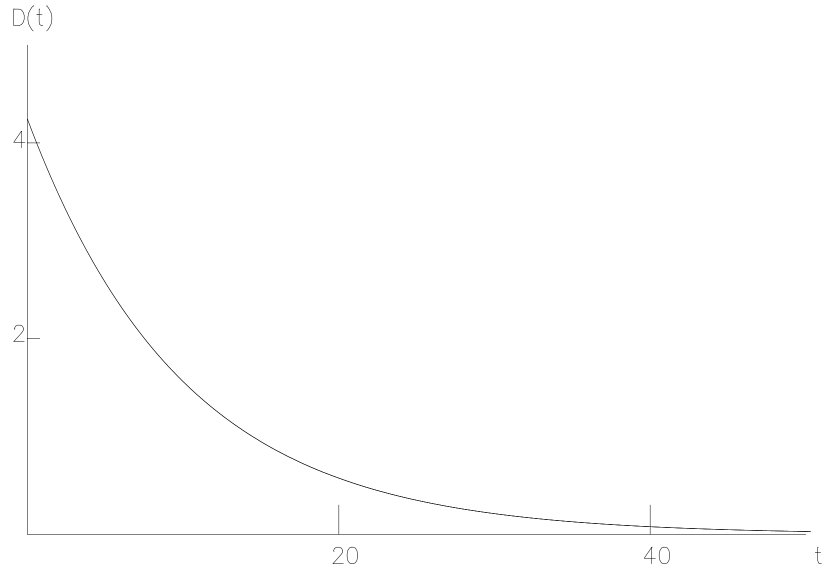

Fig. 3. Plot of the distance $D(t)$ (see (12))

of the quantity

$$D\,(t) = |c_3\,(t) - \gamma_+ \exp\,(i\omega t)| \quad \text{with} \quad c_3\,(t) = -z_1\,(t)\ z_2\,(t)\ z_3\,(t) \qquad (12)$$

(Fig. 3) that clearly provides a measure of the distance of this solution $\underline{z}\,(t)$ from its periodic limit $\underline{\tilde{z}}\,(t)$ (see (8c) and (11), as well as (1)). The numerical integration has been performed with an embedded Runge-Kutta method of order 8(5,3) with automatic step size control, as developed by Prince and Dormand [7]; the integration and the graphical output have been performed with the software Dynamics Solver developed by J. Aguirregabiria.[1] The results displayed have been obtained by integrating numerically the system of ODEs (5), checking throughout the integration the validity of the conditions (5c) as well as the two conditions

$$z_1\,(t) + z_2\,(t) + z_3\,(t) = 0\ , \quad z_1\,(t)\ z_2\,(t) + z_2\,(t)\ z_3\,(t) + z_3\,(t)\ z_1\,(t) = -3 \quad (13)$$

(see (8c)). The results reported are just a representative example of several numerical computations we did with different parameters and initial data, computations which were found to be quite reliable and stable unless the time evolution entailed a near collision of particles or their passage close to the special values $z = \pm 1$ (see (5a)).

[1] This software is available at `http://tp.lc.ehu.es/jma/ds/ds.html`

3 A (generally nonintegrable) class of asymptotically isochronous many-body models

In this section we consider a class of *asymptotically isochronous* models obtained by deforming the well-known *integrable* N-body problem with two-body inverse cube forces and a one-body linear force, which is of course *isochronous* when no deformation is present [4]. In particular we focus on the following equations of motion:

$$\ddot{x}_n + \frac{1}{4}\omega^2 x_n = g^2 \sum_{m=1, m\neq n}^{N} (x_n - x_m)^{-3} + F\left(w, \underline{x}, \underline{\dot{x}}\right) \ , \quad n = 1, ..., N \ , \tag{14a}$$

$$\dot{w} = w \left[\alpha \log w - f\left(w, \underline{x}, \underline{\dot{x}}\right)\right] \ , \tag{14b}$$

with

$$0 < w(0) < 1 \ . \tag{14c}$$

Here N is an arbitrary positive integer ($N \geq 2$); the N dependent variables $x_n \equiv x_n(t)$ may be interpreted as the coordinates of N particles evolving according to the Newtonian ("acceleration equal force") equations of motion (14a); these variables x_n are hereafter assumed to be all *real* (until we mention below to what extend the results change if the variables x_n are allowed to be *complex*), and $\underline{x}$ denotes of course the N-vector with components x_n (this has motivated the notational replacement of the particle coordinates z_n with x_n, to be kept in mind when comparing the formulas written in this section with those written in the preceeding sections); likewise the auxiliary dependent variable $w \equiv w(t)$ evolves according to the first-order ODE (14b) with an initial condition satisfying the inequalities (14c) (but clearly, see below, one could replace this first-order ODE with an *appropriate* second-order "Newtonian" ODE); t denotes of course the (*real*) independent variable ("time": ranging from the *initial* time $t = 0$ to the *asymptotic* time $t = +\infty$), and superimposed dots denote again differentiations with respect to this variable; ω, g^2 and α are three *positive* (but otherwise *arbitrary*) constants; the main restriction on the, otherwise *arbitrary*, function $F(w, \underline{x}, \underline{v})$ is that it vanish when w vanishes,

$$F\left(0, \underline{x}, \underline{v}\right) = 0 \ , \tag{15a}$$

and the main restrictions on the function $f(w, \underline{x}, \underline{v})$ is that it entail via (14b) a (very fast: see below) asymptotic vanishing (as $t \to \infty$) of the auxiliary variable $w(t)$,

$$\lim_{t\to+\infty} \left[w(t)\right] = 0 \ . \tag{16a}$$

A condition generally sufficient (but by no means necessary) to cause this is clearly (see (14b) with (14c) and below) the requirement that $f(w, \underline{x}, \underline{v})$ be *finite* and *nonnegative*,

$$0 \leq f\left(w, \underline{x}, \underline{v}\right) \leq a^2 \ , \tag{16b}$$

for all (*real*) values of w, $\underline{x}$ and $\underline{v}$; it is indeed plain (for a proof, see below) that these conditions together with (14b) entail the inequalities

$$0 < w(t) \leq [w(0)]^{\exp(\alpha t)} \quad , \tag{16c}$$

hence (see (14c) and recall that $\alpha > 0$) the auxiliary variable $w(t)$ is always positive and vanishes asymptotically *faster than exponentially*,

$$\lim_{t \to +\infty} [w(t) \exp(bt)] = 0 \; , \tag{16d}$$

with b any arbitrary constant. Restrictions on the dependence of the function $F(w, \underline{x}, \underline{v})$ upon the N-vectors $\underline{x}$ and $\underline{v}$ are also required: a simple sufficient (but of course not necessary) condition, also encompassing (15a), is that there exist a finite (*positive*) constant C and a *positive* number β such that

$$|F(w, \underline{x}, \underline{v})| \leq C\,|w|^{\beta} \quad , \quad \beta > 0 \; , \tag{17}$$

for all (*real*) values of w, $\underline{x}$ and $\underline{v}$. Functions satisfying these conditions are for instance

$$F(w, \underline{x}, \underline{v}) = C w^{\beta} \left[1 + \sum_{n=1}^{N} \left(A_n^2\, x_n^2 + B_n^2\, v_n^2\right)\right]^{-1}$$

$$F(w, \underline{x}, \underline{v}) = C w^{\beta} \exp\left[- \sum_{n=1}^{N} \left(A_n^2\, x_n^2 + B_n^2\, v_n^2\right)\right]$$

where A_n and B_n are arbitrary *real* constants.

Our main result states that, for *every* (N-vector) solution $\underline{x}(t)$ of this dynamical system, an (N-vector) $\underline{\tilde{x}}(t)$ characterizing its asymptotic behavior (as $t \to +\infty$) via the formula (1a) (exists and) has the property to be *completely periodic* (i. e., *periodic* with the *same* period in *all* its components), see (1b) with $\tilde{T} = T$, see (4a). Of course this asymptotic N-vector $\underline{\tilde{x}}(t)$ will depend on the solution $\underline{x}(t)$ under consideration – in particular, it will depend on the initial data, $\underline{x}(0)$ and $\underline{\dot{x}}(0)$, determining that solution in the context of the initial-value problem for the N-body problem (14): but let us re-emphasize that, for any arbitrary choice of these data (of course, satisfying the condition $x_n(0) \neq x_m(0)$ for $n \neq m$, see (14a)) it shall feature the property (1), namely *all* solutions $\underline{x}(t)$ of the system (14) shall feature the property of *completely isochronous asymptotic periodicity* (1) (with $\tilde{T} = T$, see (4a)).

This result is a natural consequence of the well-known fact (see for instance [4]) that *all* solutions of the system of Newtonian equations (14a) *without* the F term in the right-hand side are *completely periodic* with period T, see (4a), namely they *all* feature themselves the property (1b) with $\tilde{T} = T$. It stands therefore to reason that, if the function $F(w, \underline{x}, \underline{v})$ vanishes when w vanishes, see (15a), and if the time evolution (14b) of the auxiliary variable $w(t)$ entails that this dependent variable indeed vanishes asymptotically, see (16a), fast enough (see (16d)), then *asymptotically* all solutions of our model (14) shall behave as the solutions of the same model *without* the F term, entailing the *asymptotic* phenomenology (1) with $\tilde{T} = T$, see (4a).

To turn this hunch into a theorem a *proof* must be provided. This we do in the following subsection. Then in Section 4 we tersely discuss, again in the same qualitative vein as done above, to what extent the phenomenology described in this paper, and shown to occur in a specific, representative model, can be expected to occur in more general contexts.

3.1 A theorem and its proof

Theorem. The conditions (16b) and (17) are sufficient to guarantee that *every* solution of the N-body problem (14) with the three constants ω, g^2 and α all *positive* yield the outcomes (16a) and (1) with $\tilde{T} = T$, see (4a); in particular they guarantee that there exists, corresponding to *every* solution $\underline{x}(t)$ of the N-body problem (14), an N-vector $\underline{\tilde{x}}(t)$ satisfying both formulas (1) (of course, with z_n replaced by x_n and $\tilde{z}_n$ by $\tilde{x}_n$).

Proof. First of all let us prove the inequalities (16c), obvious as they are. To this end we set

$$w(t) = [w(0)]^{\exp[\varphi(t)]} \ , \tag{18a}$$

so that

$$\varphi(0) = 0 \tag{18b}$$

and (from (14b))

$$\dot{\varphi}(t) = \alpha + f\left[w(t), \underline{x}(t), \underline{\dot{x}}(t)\right] \exp\left[-\varphi(t)\right] \left|\log\left[w(0)\right]\right|^{-1} \ , \tag{18c}$$

where we used the fact that $\log[w(0)] = -\left|\log[w(0)]\right|$, see (14c). This ODE, together with the initial datum (18b) and the inequalities (16b), clearly imply that $\varphi(t)$ is *positive* and *finite* for $0 \leq t < \infty$, indeed validity of the inequalities

$$\alpha t < \varphi(t) < \infty \ , \quad 0 \leq t < \infty \ , \tag{18d}$$

which, via (18a) and (14c), yield (16c).

Next, let us introduce the counterpart of the Newtonian equations of motion (14a), but without the F term in the right-hand side:

$$\ddot{\tilde{x}}_n + \frac{1}{4}\omega^2 \tilde{x}_n = g^2 \sum_{m=1, m\neq n}^{N} (\tilde{x}_n - \tilde{x}_m)^{-3} \ , \quad n = 1, ..., N \ . \tag{19}$$

Here it is justified to use the notation $\tilde{x}_n \equiv \tilde{x}_n(t)$ for the dependent variables, since it is well-known [4] that *all* the solutions of this Newtonian N-body problem are *completely periodic* with period T, see (4a), consistently with (1b) with $\tilde{T} = T$.

Let us now remark that, due to the strict positivity of g^2, this system of ODEs entails that

$$|\tilde{x}_n(t) - \tilde{x}_m(t)| > \tilde{c}^2 , \quad \tilde{c}^2 > 0 , \quad n \neq m , \quad 0 \leq t < \infty , \tag{20a}$$

where $\tilde{c}^2$ is a time-independent constant that generally depends on the particular solution under consideration but is certainly strictly positive, $\tilde{c}^2 > 0$. Likewise, again due to the strict positivity of g^2, the system of ODEs (14a) with (17) and (16c) (entailing $|F(w, \underline{x}, \underline{v})| \leq D, \ D = C\,|w(0)|^\beta$) implies that

$$|x_n(t) - x_m(t)| > c^2 , \quad c^2 > 0 , \quad n \neq m , \quad 0 \leq t < \infty , \tag{20b}$$

where c^2 is again a time-independent constant that generally depends on the particular solution under consideration but is certainly strictly positive, $c^2 > 0$. Moreover the systems of ODEs (19) and (14) with (17) and (16c) clearly imply that, for all (finite, positive) time, the functions $\tilde{x}_n(t)$ and $x_n(t)$ are finite.

Let us now set

$$\xi_n(t) = x_n(t) - \tilde{x}_n(t) . \tag{21}$$

These functions $\xi_n(t)$ satisfy – as implied by subtracting (19) from (14a) – the system of ODEs

$$\ddot{\xi}_n + \frac{1}{4}\omega^2\,\xi_n + g^2 \sum_{m=1, m\neq n}^{N} [\xi_n - \xi_m]\,\varphi_{nm}(\underline{x}, \underline{\tilde{x}}) = F[w, \underline{x}, \underline{\dot{x}}] \tag{22a}$$

with

$$\varphi_{nm}(\underline{x}, \underline{\tilde{x}}) = \frac{(x_n - x_m)^2 + (x_n - x_m)(\tilde{x}_n - \tilde{x}_m) + (\tilde{x}_n - \tilde{x}_m)^2}{(x_n - x_m)^3\,(\tilde{x}_n - \tilde{x}_m)^3} . \tag{22b}$$

Note that the above bounds, (20), as well as the finiteness of x_n and $\tilde{x}_n$ for all (positive) time, guarantee that these functions $\varphi_{nm}(\underline{x}, \underline{\tilde{x}})$ remain *finite* for all time, namely that there always exist time-independent *finite* upper and lower bounds $\varphi_\pm$ satisfied by them for all time,

$$\varphi_- \leq \varphi_{nm}(\underline{x}, \underline{\tilde{x}}) \leq \varphi_+ . \tag{22c}$$

These bounds depend of course on the particular solutions $\underline{x}$ and $\underline{\tilde{x}}$ under consideration, but let us re-emphasize that, for any such solutions, they are *finite*.

It is now clear that the theorem is proven if we can show that this system of ODEs admits a solution satisfying the asymptotic condition

$$\lim_{t \to +\infty} \left[\xi_n \left(t \right) \right] = 0 \ , \quad n = 1, ..., N \tag{23}$$

(see (1a) and (21)). As can be easily verified such a solution of (22) is provided by the formula

$$\xi_n \left(t \right) = \int_t^{\infty} dt' F \left[w \left(t' \right), \underline{x} \left(t' \right), \underline{\dot{x}} \left(t' \right) \right] \, G_n \left(t, t' \right) \ , \quad n = 1, ..., N \ , \tag{24a}$$

where the functions $G_n \left(t, t' \right)$ are the Green's functions associated with the left-hand side of the system of ODEs (22a), namely the solutions of the system of ODEs

$$\frac{\partial^2 G_n \left(t, t' \right)}{\partial t^2} + \frac{1}{4} \omega^2 \, G_n \left(t, t' \right)$$
$$+ g^2 \sum_{m=1, m \neq n}^{N} \left[G_n \left(t, t' \right) - G_m \left(t, t' \right) \right] \, \varphi_{nm} \left[\underline{x} \left(t \right), \underline{\tilde{x}} \left(t \right) \right] = 0 \ , \quad t \leq t' \ ,$$
$$G_n \left(t, t \right) = 0 \ , \quad \left. \frac{\partial G_n \left(t, t' \right)}{\partial t} \right|_{t = t'} = -1 \ , \quad n = 1, ..., N \ . \tag{24b}$$

Indeed, while these Green functions cannot be computed explicitly (since we do not know the N-vectors $\underline{x} \left(t \right)$ and $\underline{\tilde{x}} \left(t \right)$, hence neither the functions $\varphi_{nm} \left[\underline{x} \left(t \right), \underline{\tilde{x}} \left(t \right) \right]$), it is plain from the linear character of this system of ODEs and from the bounds (22c) that these Green functions can grow (in modulus) at most exponentially as $t \to \infty$ and/or $t' \to \infty$; so that the *faster than exponential* vanishing of $F \left[w \left(t' \right), \underline{x} \left(t' \right), \underline{\dot{x}} \left(t' \right) \right]$ as $t' \to \infty$ (implied by (17) with (16d)) entails that the integral in the right-hand side of the solution formula (24a) vanishes asymptotically (as $t \to \infty$). □

Remark. It is clear how this example could have been made more general by allowing the function F appearing in the right hand side of (14a) to depend on the index n, and/or by replacing the single auxiliary variable $w \left(t \right)$ by a J-vector $\underline{w} \left(t \right)$ with J an arbitrary positive integer, and so on; without invalidating our conclusion, but complicating our proof. Let us also re-emphasize that the hypotheses made above to prove this *theorem* are *sufficient* but by no means *necessary* for its validity. More specific, and possibly considerably less stringent, conditions yielding an analogous conclusion can and will be introduced whenever this kind of result shall be considered in specific (possibly applicative) contexts. Our motivation to assume here quite simple (hence overly stringent) hypotheses is because we are just interested to show that the main idea discussed in this paper does indeed work. □

4 Outlook

Clearly the kind of approaches illustrated above via the detailed treatment of two specific examples can be applied much more widely: it will be particularly interesting to do so in specific applicative contexts.

A natural point of departure for such applications are *isochronous* systems, namely models whose *generic* solutions – in their *entire* natural phase space, or in *open*, hence fully dimensional, regions of it – are *completely periodic* (i. e., periodic in *all* their degrees of freedom) with the *same fixed* period (independent of the initial data, provided they stay within the *isochronicity* region). As recently pointed out (see for instance [2]), quite a lot of dynamical systems can be modified so that they become *isochronous*, entailing the conclusion that *isochronous systems are not rare*. Each of these *isochronous* systems can then be further extended – along the lines obviously suggested by the treatment detailed above, see in particular the specific case treated in Section 3 – in order to generate classes of *asymptotically isochronous* systems, namely systems featuring *open*, hence fully dimensional, regions in their natural phase space (possibly including all of it) in which *all* (or *almost all*) their solutions display asymptotically a *completely periodic* behavior with the *same fixed* period, see (1). The technique to manufacture such generalized systems is clearly suggested by the examples treated above: of course these systems could be *autonomous,* as the examples treated above, or they might feature an *explicit* time-dependence, as could have been included in the system treated in Section 3 by assuming the functions F and f to also feature an *explicit* time dependence (but *autonomous* systems are generally more interesting than *nonautonomous* ones).

Often the natural context to investigate *isochronous* systems is in the *complex* rather than the *real* [2, 4] – although every system with *complex* dependent variables can of course be reformulated as a system with twice as many *real* dependent variables. Hence it may be of interest to mention how the findings detailed in Section 3 would be affected if the dependent variables x_n and w in the model (14) were allowed to be *complex* – keeping of course *real* the time t and *positive* the constant ω, while the constant g^2 could now also be *complex.* It is then well known [2, 4] that the *isochronous* character of the motions still prevails for the (*integrable* indeed *solvable*) many-body problem (14a) *without* the F term (i. e., with an identically vanishing F; see (19)) – describing motions taking place in the *complex* z-plane rather than on the *real* line. But in the *complex* context the *isochronous* behavior is a bit different than in the *real* context: the phase space is then divided into sectors separated by lower-dimensional manifolds characterized by solutions which hit a *singularity* at a finite time due to a particle collision; an event forbidden in the *real* case with *positive* g^2, when the particles move on the *real* axis and the two-body force, *singular* at zero separation, is *repulsive*, see (14a), but which can happen in the *complex* case, although not for *generic* initial data. In the different sectors the motion is still *completely periodic*, but with different periods, characterizing each sector and being (generally rather small [6]) *integer* multiples of the basic period T, see (4a). Accordingly, the *generic* solution of the (generally *nonintegrable*) generalized model (14) will be *nonsingular* throughout its time evolution and it shall eventually settle within a sector, approaching asymptotically one of the *completely periodic* solutions in that sector of the (*integrable*) model (14a) with identically vanishing F.

A somewhat analogous outcome obtains for the model analogous to (14) but with (14a) replaced by

$$\ddot{z}_n + \frac{1}{4}\omega^2 z_n = \sum_{m=1, m\neq n}^{N} \left[g_{nm}^2 \left(z_n - z_m \right)^{-3} \right] + F\left(w, \underline{z}, \underline{\dot{z}}\right) \ , \quad n = 1, ..., N \ , \quad (25)$$

featuring $N\left(N-1\right)$ different coupling constants g_{nm}^2 acting among every particle pair. In this case the model without F is generally *not integrable*, yet (if considered in the *complex*, namely without restricting the dependent variables z_n – nor, for that matter, the coupling constants g_{nm}^2 – to be *real*) it still does feature an *open*, hence fully dimensional, region in its phase space where *all* solutions are *completely periodic* with the same period T, see (4a) [2, 8]; while in other regions of its phase space it might also be *periodic* but with periods $\tilde{T} = pT$ where the numbers p are *integers* but might be very large, or it might even display an *aperiodic*, quite complicated (in some sense *chaotic*) behavior [9] (for recent progress in the understanding of this phenomenology see [10–13]). It then stands to reason that the solutions of the generalized model (14) with (14a) replaced by (25) (and of course $\underline{x}$ in (14b) replaced by $\underline{z}$) shall again approach asymptotically solutions – including, from *open* regions of initial data, *completely periodic* ones – of the model (25) *without* F: entailing a remarkable, and quite rich, phenomenology. Clearly our motivation to mention this specific model is because of its prototypical role: indeed, the main aspects of this phenomenology shall also characterize the large class of *isochronous* (but by no means necessarily *integrable*) systems that can now be manufactured [2], once they are extended by adding to their equations of motion other, fairly general, terms having the property to disappear asymptotically (as $t \to +\infty$), as a consequence of the very dynamics implied by these extended equations of motion.

In conclusion let us re-emphasize that these results (as indeed all mathematically correct findings) might well be deemed remarkable or trivial, depending on the level of understanding of the reader. Once their foundation is understood, it becomes obvious how they can be extended to many other models – suggesting an ample applicative potential. But these developments exceed the scope of this paper.

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